On the boundedness of certain elliptic operators in generalized Morrey spaces (Harmonic Analysis and Nonlinear Partial Differential Equations)

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By

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Abstract

In this paper we prove the boundedness of Bessel potential operator on the generalized Morrey space $\mathcal{M}_p^{\varphi}(\Omega)$, where Ω is a bounded smooth domain in $\mathbb{R}^n$. We also establish the boundedness of certain elliptic operators on these spaces. To obtain these results, we utilize Gaffney type estimates for these operators on Lebesgue spaces.

§ 1. Introduction

Let $\nu > 0$ be a constant. In this paper, we shall discuss the Bessel potential operator $(1-\nu\Delta)^{-1}$ which is defined by

$$(1-\nu\Delta)^{-1}f(x) := \int_{\Omega} K(x-y)f(y)\, dy \ (x \in \Omega),$$

where $f \in L^p(\mathbb{R}^n)$ for $p \geq 1$ and the kernel K is defined by

$$K(x) := \frac{1}{4\nu^{n/2}\pi} \int_0^{\infty} e^{-\frac{\pi|x|^2}{\nu\delta} - \frac{\delta}{4\pi}} \delta^{-\frac{n}{2}}\, d\delta \ (x \in \mathbb{R}^n). \tag{1.1}$$

Remark that this operator is related to the equation

$$u - \nu\Delta u = f \tag{1.2}$$

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in $\mathbb{R}^n$. Indeed, $u = (1-\nu\Delta)^{-1}f$ is a solution to (1.2) (see [7, p. 186]). It is known that the operator $(1-\nu\Delta)^{-1}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 \le p \le \infty$ and

$$\|(1-\nu\Delta)^{-1}f\|_{L^p(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)} \tag{1.3}$$

for every $f \in L^p(\mathbb{R}^n)$; see [17, p. 135] for instance. In this paper, we prove the boundedness of $(1-\nu\Delta)^{-1}$ on the generalized Morrey spaces $\mathcal{M}_p^\varphi(\Omega)$, where Ω is a bounded smooth domain in $\mathbb{R}^n$. This function space is introduced in [13]. Let us recall the definition of $\mathcal{M}_p^\varphi(\Omega)$.

Definition 1.1. Let $1 \le p < \infty$ and $\varphi : (0,\infty) \to (0,\infty)$. The generalized Morrey space $\mathcal{M}_p^\varphi(\Omega)$ is defined to be the space of all functions $f \in L^p(\Omega)$ for which

$$\|f\|_{\mathcal{M}_p^\varphi(\Omega)} := \sup_{a\in\Omega, 0<r<\operatorname{diam}(\Omega)} \varphi(r) \left(\frac{1}{|B(a,r)|}\int_{\Omega\cap B(a,r)} |f(x)|^p \, dx\right)^{\frac{1}{p}} < \infty,$$

where $B(a,r)$ denotes the ball centered at a and of radius r.

Note that, for $\varphi(r) := r^{\frac{n}{p}}$, we have $\mathcal{M}_p^\varphi(\Omega) = L^p(\Omega)$. By letting $\varphi(r) := r^{\frac{n-\lambda}{p}}$ where $0 \le \lambda \le n$, we can recover the (classical) Morrey spaces $L^{p,\lambda}(\Omega)$, introduced in [11]. In this note we also assume that $\varphi \in \mathcal{G}_p$, that is, φ is increasing and the function $t \mapsto t^{-\frac{n}{p}}\varphi(t)$ is decreasing. Now we state our main result as follows.

Theorem 1.2. *Let $n > 2$, $1 < p < \infty$ and $\varphi \in \mathcal{G}_p$. Then there exists a constant $C > 0$ such that*

$$\|(1-\nu\Delta)^{-1}f\|_{\mathcal{M}_p^\varphi(\Omega)} \le C\|f\|_{\mathcal{M}_p^\varphi(\Omega)} \tag{1.4}$$

for every $f \in \mathcal{M}_p^\varphi(\Omega)$.

Observe that we do not assume some integral type conditions for φ, which are usually assumed for the boundedness of fractional integral operators and Bessel-Riesz potential operators (see Theorems 2.9 and 2.10). We shall discuss some connections between these operators and some special cases of Theorem 1.2 in Subsection 2.2 (see Corollaries 2.11 and 2.12).

Theorem 1.2 can be seen as a model case of the boundedness of certain elliptic operators on generalized Morrey spaces. Let Ω be a bounded domain in $\mathbb{R}^n$. Let $\{a_{ij}\}_{i,j=1,\dots,n}$ be a collection of real-valued measurable functions on Ω for which $a_{ij} = a_{ji}$ and

$$\lambda^{-1}|w|^2 \le \sum_{i,j=1}^n a_{ij}(x)w_iw_j \le \lambda|w|^2 \ (\text{a.e.}x \in \Omega), \tag{1.5}$$

for some constant $\lambda > 1$ and for every $w \in \mathbb{R}^n$. Let us consider the operator

$$Lu := -\sum_{i,j=1}^{n} D_j(a_{ij}D_i u),$$

where u belongs to the Sobolev space $W_0^{1,p}(\Omega)$ and $D_i u$ denotes the weak derivative of u with respect to x_i. For example, if $a_{ij} = \begin{cases} 1, & i = j, \\ 0 & i \neq j \end{cases}$, then $Lu = -\Delta u$. Recall that, by virtue of the Riesz representation theorem, for $f \in L^2(\Omega)$, the equation

$$Lu = f \tag{1.6}$$

has a weak solution $u \in W_0^{1,2}(\Omega)$ and the L^2-norm of Du is dominated by the L^2-norm of f (see [7, chapter 6]). The case $p \neq 2$, continuous coefficients a_{ij} and $f = \mathrm{div}(g)$ are discussed in [4, 12, 16]. The study of this L^p-estimate for discontinuous coefficients can be seen in [2, 3, 6].

In this paper we discuss the following variant of (1.6):

$$u + \nu Lu = f, \tag{1.7}$$

where $\nu > 0$, $f : \Omega \to \mathbb{R}$ is an integrable function on Ω, and $u : \Omega \to \mathbb{R}$ is the unknown. Recall that $u \in W_0^{1,p}(\Omega)$, $1 < p < \infty$, is called a weak solution of (1.7) if

$$\int_\Omega \left(ug + \nu \sum_{i,j=1}^{n} a_{ij} D_i u D_j g \right) dx = \int_\Omega fg \, dx,$$

for every $g \in W_0^{1,p'}(\Omega)$, where $\frac{1}{p'} := 1 - \frac{1}{p}$. For $p = 2$, by an application of Lax-Milgram theorem, we know that there exists a unique weak solution u of (1.7). Moreover,

$$\|Du\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}, \tag{1.8}$$

for some $C > 0$ and for every $f \in L^2(\Omega)$. In this case, we define the operator $(1+\nu L)^{-1} : L^2(\Omega) \to W_0^{1,2}(\Omega)$ by

$$(1 + \nu L)^{-1} f := u.$$

Our second result is an extension of (1.8) to the generalized Morrey space $\mathcal{M}_2^\varphi(\Omega)$.

Theorem 1.3. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$. Assume that $\varphi \in \mathcal{G}_2$ and there exists a constant $C_0 > 0$ such that*

$$\int_r^\infty \frac{t^{n/2-1}}{\varphi(t)} e^{-t} \, dt \leq C_0 \frac{r^{n/2}}{\varphi(r)}, \tag{1.9}$$

for every $r > 0$. Then there exists a constant $C > 0$ such that

$$\|D((1+\nu L)^{-1}f)\|_{\mathcal{M}_2^{\varphi}(\Omega)} \le C\|f\|_{\mathcal{M}_2^{\varphi}(\Omega)}, \tag{1.10}$$

for every $f \in \mathcal{M}_2^{\varphi}(\Omega)$.

Note that we can recover (1.8) from Theorem 1.3 by taking $\varphi(t) := t^{n/2}$. For the case $p > 2$, we assume that the coefficients a_{ij} belong to the class vanishing mean oscillation (VMO) (see [14] and Definition 2.6).

Theorem 1.4. *Let $n \ge 3$ and Ω be a bounded smooth domain in $\mathbb{R}^n$. Assume that the coefficients $a_{ij} \in \mathrm{VMO}(\Omega)$. Let $2 < p < \infty$ and $\varphi \in \mathcal{G}_p$. Assume that there exists a constant $C_0 > 0$ such that*

$$\int_r^{\infty} \frac{t^{n/p-1}}{\varphi(t)} e^{-t}\, dt \le C_0 \frac{r^{n/p}}{\varphi(r)}, \tag{1.11}$$

for every $r > 0$. Then there exists a constant $C > 0$ such that

$$\|D((1+\nu L)^{-1}f)\|_{\mathcal{M}_p^{\varphi}(\Omega)} \le C\|f\|_{\mathcal{M}_p^{\varphi}(\Omega)}, \tag{1.12}$$

for every $f \in \mathcal{M}_p^{\varphi}(\Omega)$.

Observe that the inequality (1.11) is trivial for $\varphi(t) := t^{n/p}$. Therefore, we think that the condition (1.11) is too strong. Based on this observation and Theorem 1.2, we conjecture that (1.12) holds without assuming (1.11).

Conjecture 1.5. *Let $n \ge 3$ and Ω be a bounded smooth domain in $\mathbb{R}^n$. Assume that the coefficients $a_{ij} \in \mathrm{VMO}(\Omega)$. Let $2 \le p < \infty$, $\varphi \in \mathcal{G}_p$, and $f \in \mathcal{M}_p^{\varphi}(\Omega)$. Then the inequality* (1.12) *holds.*

The organization of our paper is as follows. In Section 2, we recall some definitions and notation for some integral operators and function spaces in this paper. We also state some known results for the boundedness of these integral operators. In Section 3, we prove Gaffney type estimates for Bessel potential operator on Lebesgue spaces. The proof of Theorem 1.2 will be given in Section 4. We shall prove Theorems 1.3 and 1.4 in the last section. Finally, we set our notation as follows. The notation $X \lesssim Y$ means there exists a positive constant C such that $X \le CY$ and C is independent of appropriate quantities. We write $X \sim Y$ if and only if $X \lesssim Y$ and $Y \lesssim X$. For $1 \le p \le \infty$, we always define p' by $\frac{1}{p'} := 1 - \frac{1}{p}$.

§ 2. Preliminaries

In this section we recall some definitions and notation of some function spaces and integral operators in this paper. We also recall the boundedness results for these operators on Morrey spaces. In the last subsection we also recall and prove some basic estimates for the Bessel kernel and the Bessel potential operator.

§ 2.1. Some definitions and notation

First we recall the definition of Sobolev spaces.

Definition 2.1. Let $1 \leq p < \infty$ and $\Omega \subseteq \mathbb{R}^n$. The Sobolev space $W^{1,p}(\Omega)$ is defined to be the set of all locally integrable functions $u : \Omega \to \mathbb{R}$ such that u and its weak derivatives of order 1 belong to $L^p(\Omega)$.

We also define $W_0^{1,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{W^{1,p}(\Omega)}$ where $C_c^\infty(\Omega)$ is the set of all smooth functions with compact support.

In this article we shall use some properties of the Hardy-Littlewood maximal operator and fractional integral operators whose definitions are given as follows.

Definition 2.2. The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy, \ (x \in \mathbb{R}^n),$$

for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Definition 2.3. Let $0 < \alpha < n$. The fractional integral operator I_α is defined by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \ (x \in \mathbb{R}^n),$$

for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$

The operator I_α can be seen as a special case of Bessel-Riesz operator which is defined as follows (see [9] and references therein).

Definition 2.4. Let $0 < \alpha < n$ and $\gamma \geq 0$. The Bessel-Riesz operator $I_{\alpha,\gamma}$ is defined by the formula

$$I_{\alpha,\gamma} f(x) := K_{\alpha,\gamma} * f(x), \ (x \in \mathbb{R}^n),$$

for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, where $K_{\alpha,\gamma}(x) := \frac{1}{|x|^{n-\alpha}(1+|x|)^\gamma}$.

In Theorem 1.4, we assume that the coefficients a_{ij} belong to the class vanishing mean oscillation (VMO). This space is a subspace of bounded mean oscillation (BMO) space, introduced in [10]. We recall the definition of these spaces as follows.

Definition 2.5. [10] Let $x \in \mathbb{R}^n$, $r > 0$, and $f \in L^1_{\rm loc}(\mathbb{R}^n)$. Define

$$f_{B(x,r)} := \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\ dy.$$

and

$$\eta_f(r) := \sup_{x\in\mathbb{R}^n, \rho\le r} \frac{1}{|B(x,\rho)|}\int_{B(x,\rho)} |f(y) - f_{B(x,\rho)}|\ dy.$$

The space $\mathrm{BMO}(\mathbb{R}^n)$ is the set all functions $f \in L^1_{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{\rm BMO} := \sup_{r>0} \eta_f(r) < \infty. \tag{2.1}$$

Definition 2.6. [14] If $f \in \mathrm{BMO}(\mathbb{R}^n)$ and moreover satisfies

$$\lim_{r\to 0^+} \eta_f(r) = 0,$$

then we say that $f \in \mathrm{VMO}(\mathbb{R}^n)$. For a bounded domain Ω in $\mathbb{R}^n$, we define

$$\mathrm{BMO}(\Omega) := \{f \in L^1_{\rm loc}(\Omega)\ :\ f = b|\Omega \text{ for some } b \in \mathrm{BMO}(\mathbb{R}^n)\}$$
$$\mathrm{VMO}(\Omega) := \{f \in L^1_{\rm loc}(\Omega)\ :\ f = b|\Omega \text{ for some } b \in \mathrm{VMO}(\mathbb{R}^n)\}.$$

§ 2.2. The boundedness of some integral operators on generalized Morrey spaces

First let us recall the following pointwise estimate of $I_\alpha f(x)$, which is known as Hedberg's inequality.

Lemma 2.7. [8, p. 2] *Let $0 < \alpha < n$, $1 \le p < \frac{n}{\alpha}$, and $\frac{1}{q} := \frac{1}{p} - \frac{\alpha}{n}$. Then*

$$|I_\alpha f(x)| \lesssim Mf(x)^{\frac{p}{q}} \|f\|_{L^p}^{1-\frac{p}{q}}$$

for every $f \in L^p(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

Next we recall the result on the boundedness of the Hardy-Littlewood maximal operator on $\mathcal{M}^\varphi_p(\Omega)$.

Theorem 2.8. [13, 15] *Let $1 < p < \infty$ and assume that $\varphi \in \mathcal{G}_p$. Then*

$$\|Mf\|_{\mathcal{M}^\varphi_p(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{M}^\varphi_p(\mathbb{R}^n)} \tag{2.2}$$

for every $f \in \mathcal{M}^\varphi_p(\mathbb{R}^n)$.

We remark that Theorem 2.8 was first proved in [13] by assuming that

$$\int_r^\infty \frac{1}{t\varphi(t)}\,dt \lesssim \frac{1}{\varphi(r)}, \tag{2.3}$$

for every $r > 0$. This assumption is removed in [15]. Now we recall the boundedness of I_α and $I_{\alpha,\gamma}$ in generalized Morrey spaces and comparing these results with Theorem 1.2.

Theorem 2.9. [13] *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\varphi \in \mathcal{G}_p$, $\frac{1}{q} := \frac{1}{p} - \frac{\alpha}{n}$, and $\psi(t) := t^{-\alpha}\varphi(t)$. Assume that*

$$\int_r^\infty \frac{t^{\alpha-1}}{\varphi(t)}\,dt \lesssim \frac{1}{\psi(r)},$$

for every $r > 0$. Then I_α is bounded from $\mathcal{M}_p^\varphi(\mathbb{R}^n)$ to $\mathcal{M}_p^\psi(\mathbb{R}^n)$.

Theorem 2.10. [9] *Let $0 < \alpha < n$, $\gamma > 0$, $1 \le p < \frac{n}{\alpha}$, $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$, $1 \le s \le t$, and $\varphi \in \mathcal{G}_p$ satisfy*

$$\int_R^\infty \frac{r^{\frac{n}{t'}-1}}{\varphi(r)}\,dr \lesssim \frac{R^{\frac{n}{t'}}}{\varphi(R)},$$

for every $R > 0$. Define $\frac{1}{q} := \frac{1}{p} - \frac{1}{s'}$ and $\psi(r) := \frac{\varphi(r)}{r^{\frac{n}{t'}}}$. Then

$$\|I_{\alpha,\gamma} f\|_{\mathcal{M}_q^\psi(\mathbb{R}^n)} \lesssim \|K_{\alpha,\gamma}\|_{\mathcal{M}_t^s(\mathbb{R}^n)} \|f\|_{\mathcal{M}_p^\varphi(\mathbb{R}^n)}$$

for every $f \in \mathcal{M}_p^\varphi(\mathbb{R}^n)$.

Remark that one may replace $\mathbb{R}^n$ by Ω in Theorems 2.9 and 2.10. Moreover, by using these theorems and embedding of Morrey spaces, we obtain the following special cases of Theorem 1.2.

Corollary 2.11. *Let $n > 2$, $1 < p < \frac{n}{2}$, and $\varphi \in \mathcal{G}_p$ satisfy $\int_r^\infty \frac{t}{\varphi(t)}\,dt \lesssim \frac{r^2}{\varphi(r)}$. Then*

$$\|(1-\nu\Delta)^{-1} f\|_{\mathcal{M}_p^\varphi(\Omega)} \lesssim \|f\|_{\mathcal{M}_p^\varphi(\Omega)}, \tag{2.4}$$

for every $f \in \mathcal{M}_p^\varphi(\Omega)$.

Proof. Let $\frac{1}{q} := \frac{1}{p} - \frac{2}{n}$ and $\psi(r) := r^{-2}\varphi(r)$. Then (2.4) follows from the inequality

$$|(1-\nu\Delta)^{-1} f(x)| \le I_2(|f|)(x),$$

embedding $\mathcal{M}_q^\psi(\Omega) \subseteq \mathcal{M}_p^\varphi(\Omega)$, and Theorem 2.9. $\square$

Corollary 2.12. *Let $n > 2$, $1 < p < \frac{n}{2}$, and $\varphi \in \mathcal{G}_p$. Assume that*

$$\int_R^\infty \frac{r^{\frac{n}{t'}-1}}{\varphi(r)}\, dr \lesssim \frac{R^{\frac{n}{t'}}}{\varphi(R)},$$

for every $R > 0$, where $\frac{n}{n+\gamma-2} < t < \frac{n}{n-2}$ and $\gamma > 0$. Then (2.4) holds.

Proof. Let $s \in (1, t]$, $\frac{1}{q} := \frac{1}{p} - \frac{1}{s'}$, and $\psi(r) := r^{-\frac{n}{t'}}\varphi(r)$. By virtue of Theorem 2.10, embedding $\mathcal{M}_q^\psi(\Omega) \subseteq \mathcal{M}_p^\varphi(\Omega)$, and

$$|(1-\nu\Delta)^{-1}f(x)| \le I_{2,\gamma}(|f|)(x),$$

we get (2.4). □

§ 2.3. Basic estimates for Bessel potential operator

We recall the following fact about the size of the kernel K and its gradient given in [18].

Lemma 2.13. [18, p. 65] *There exists a positive constant C such that*

$$0 \le K(x) \lesssim \frac{e^{-C|x|}}{|x|^{n-2}} \text{ and } |DK(x)| \lesssim \frac{e^{-C|x|}}{|x|^{n-1}} \tag{2.5}$$

for every $x \in \mathbb{R}^n$.

We prove the following L^2-estimate of the gradient of $(1-\nu\Delta)^{-1}f$, for every $f \in L^2(\Omega)$.

Lemma 2.14. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and let E be a closed subset of Ω. For each $\nu > 0$ and $f \in L^2(\Omega)$ satisfying* $\operatorname{supp}(f) \subseteq E$, *we define*

$$u(x) := (1-\nu\Delta)^{-1}f(x) \ (x \in \mathbb{R}^n).$$

Then

$$\|\nabla u\|_{L^2(\mathbb{R}^n)} \le \frac{1}{\sqrt{\nu}}\|f\|_{L^2(E)}. \tag{2.6}$$

Proof. Suppose that $f \in C_c^\infty(\Omega)$ with $\operatorname{supp}(f) \subseteq E$. Then $u \in C^\infty(\mathbb{R}^n)$. Let B_R be any ball of radius $R \ge 2$ and centered at the origin such that $\Omega \subseteq B_{R/2}$. Integration by part yield

$$\begin{aligned}\int_{B_R} |\nabla u(x)|^2\, dx &= -\int_{B_R} u(x)\Delta u(x)\, dx + \int_{\partial B_R} u(x)\nabla u(x)\cdot \mathbf{n}(x)\, dS(x)\\ &= \frac{1}{\nu}\int_{B_R} (u(x)f(x) - u(x)^2)\, dx + \int_{\partial B_R} u(x)\nabla u(x)\cdot\mathbf{n}(x)\, dS(x),\end{aligned}$$

where $\mathbf{n}(x)$ denotes the unit normal vector on ∂B_R. For $x \in \partial B_R$, we have

$$
\begin{aligned}
|u(x)| &\le \int_\Omega |K(x-y)||f(y)|\ dy \\
&\lesssim \int_\Omega e^{-C|x-y|}|f(y)|\ dy \\
&\lesssim \int_\Omega e^{-CR}|f(y)|\ dy \lesssim \|f\|_{L^1(E)} e^{-CR},
\end{aligned}
\tag{2.7}
$$

Similarly

$$
|\nabla u(x)| \lesssim \|f\|_{L^1(E)} e^{-CR}, \tag{2.8}
$$

for every $x \in \partial B_R$. Combining (2.7) and (2.8), we get

$$
\left| \int_{\partial B_R} u(x)\nabla u(x) \cdot \mathbf{n}(x)\ dS(x) \right| \lesssim \|f\|^2_{L^1(E)} R^{n-1} e^{-CR} \to 0
$$

as $R \to \infty$. Consequently,

$$
\begin{aligned}
\int_{\mathbb{R}^n} |\nabla u(x)|^2\ dx &= \frac{1}{\nu} \lim_{R\to\infty} \int_{B_R} (f(x)u(x) - u(x)^2)\ dx \\
&= \frac{1}{\nu} \int_E f(x)u(x)\ dx - \frac{1}{\nu} \int_{\mathbb{R}^n} u(x)^2\ dx.
\end{aligned}
$$

By Cauchy's inequality, we get

$$
\int_{\mathbb{R}^n} |\nabla u(x)|^2\ dx \le \frac{1}{\nu} \int_E f(x)^2\ dx. \tag{2.9}
$$

Now suppose that $f \in L^2(\Omega)$ satisfies $\mathrm{supp}(f) \subseteq E$. Then there exists a sequence $\{f_j\}_{j=1}^\infty \subseteq C_c^\infty(\Omega)$ with $\mathrm{supp}(f_j) \subseteq E$ such that

$$
\lim_{j\to\infty} \|f - f_j\|_{L^2(E)} = 0. \tag{2.10}
$$

Define $u_j(x) := (1-\nu\Delta)^{-1} f_j(x)$ for every $x \in \mathbb{R}^n$. Then, it follows from (1.3) that

$$
\|u - u_j\|_{L^2(\mathbb{R}^n)} \le \|f - f_j\|_{L^2(E)},
$$

for every $j \in \mathbb{N}$. Consequently,

$$
\lim_{j\to\infty} \|u - u_j\|_{L^2(\mathbb{R}^n)} = 0. \tag{2.11}
$$

In particular,

$$
\|u_j - u_k\|_{L^2(\mathbb{R}^n)} \to 0 \tag{2.12}
$$

as $j,k\to\infty$. Moreover, as a consequence of (2.9) and (2.10), we have

$$\|\nabla u_j-\nabla u_k\|_{L^2(\mathbb{R}^n)}\to 0 \tag{2.13}$$

as $j,k\to\infty$. In view of (2.12) and (2.13), there exists $\tilde{u}\in W^{1,2}(\mathbb{R}^n)$ such that

$$\lim_{j\to\infty}\|\tilde{u}-u_j\|_{W^{1,2}(\mathbb{R}^n)}=0. \tag{2.14}$$

Combining (2.11) and (2.14), we see that $u=\tilde{u}$, so $u\in W^{1,2}(\mathbb{R}^n)$ and

$$\lim_{j\to\infty}\|u-u_j\|_{W^{1,2}(\mathbb{R}^n)}=0. \tag{2.15}$$

We use (2.9) again to obtain

$$\|\nabla u\|_{L^2(\mathbb{R}^n)}\le\|u-u_j\|_{W^{1,2}(\mathbb{R}^n)}+\frac{1}{\sqrt{\nu}}\|f-f_j\|_{L^2(E)}+\frac{1}{\sqrt{\nu}}\|f\|_{L^2(E)}. \tag{2.16}$$

Thus, (4.1) follows from (2.10), (2.15), and (2.16). □

§ 3. A Gaffney type estimate for $(1-\nu\Delta)^{-1}$ on $L^p(\Omega)$

In this section we prove an estimate of Gaffney type for the operator $(1-\nu\Delta)^{-1}$. The proof for the case $p=2$ is similar to that of [1, Lemma 2.1] but we give the detail for the reader's convenience.

Proposition 3.1. *Let $\nu>0$ be a constant and let Ω be a bounded smooth domain in $\mathbb{R}^n$. Let E and F be disjoint closed subsets of Ω with $d:=\mathrm{dist}(E,F)>0$. Then there exists $a>0$ such that for all $f\in L^2(\Omega)$ satisfying* $\mathrm{supp}(f)\subseteq E$

$$\|(1-\nu\Delta)^{-1}f\|_{L^2(F)}\lesssim e^{-\frac{ad}{\sqrt{\nu}}}\|f\|_{L^2(E)}. \tag{3.1}$$

Proof. Let $t:=\sqrt{\nu}$. Suppose that $t\in[d,\infty)$. By virtue of the L^2-boundedness of the operator $(1-\nu\Delta)^{-1}$, we have

$$\|(1-\nu\Delta)^{-1}f\|_{L^2(F)}\le\|(1-\nu\Delta)^{-1}f\|_{L^2(\Omega)}\le\|f\|_{L^2(E)}\le e\cdot e^{-\frac{d}{t}}\|f\|_{L^2(E)}.$$

We prove (3.1) for $0<t\le d$, which is essential. We define

$$u^t:=(1-t^2\Delta)^{-1}f \text{ and } \tilde{F}:=\{x\in\Omega:\mathrm{dist}(x,F)\le\mathrm{dist}(x,E)\}.$$

Choose $\eta\in C^\infty(\mathbb{R}^n)$ such that

$$\chi_F\le\eta\le\chi_{\tilde{F}} \text{ and } \|\nabla\eta\|_{L^\infty}\le\frac{C}{d}.$$

Set $\alpha := \frac{1}{t\|\nabla\eta\|_{L^\infty}}$. If $\alpha \leq 2$, then

$$\int_F |u^t(x)|^2\ dx \leq \int_E |f(x)|^2\ dx \leq e^{2-\alpha}\|f\|_{L^2(E)}^2 \sim e^{-\frac{1}{t\|\nabla\eta\|_{L^\infty}}}\|f\|_{L^2(E)}^2 \leq e^{-\frac{d}{Ct}}\|f\|_{L^2(E)}^2.$$

Letting $a := \frac{1}{2C}$, we get

$$\|(1-\nu\Delta)^{-1}f\|_{L^2(F)} \lesssim e^{-\frac{ad}{t}}\|f\|_{L^2(E)}.$$

Now assume that $\alpha > 2$. Define

$$\zeta(x) := e^{\alpha\eta(x)} - 1.$$

By using the inequality $e^{2\alpha} \leq 3(e^\alpha - 1)^2$, we have

$$\int_F |u^t(x)|^2\ dx \leq 3e^{-2\alpha}\int_\Omega |u^t(x)|^2\zeta(x)^2\ dx \tag{3.2}$$

Since $\mathrm{supp}(f)\cap\mathrm{supp}(\zeta) = \emptyset$, we see that

$$\begin{aligned}\int_\Omega |u^t(x)|^2\zeta(x)^2\ dx &= \int_\Omega f(x)u^t(x)\zeta(x)^2\ dx + t^2\int_\Omega \Delta u^t(x)u^t(x)\zeta(x)^2\ dx\\ &= t^2\int_\Omega \Delta u^t(x)u^t(x)\zeta(x)^2\ dx.\end{aligned}\tag{3.3}$$

Let $\varepsilon > 0$ be determined later. Integration by part and Cauchy's inequality yield

$$\begin{aligned}\int_\Omega |u^t(x)|^2\zeta(x)^2\ dx &= -t^2\int_\Omega |\nabla u^t(x)|^2\zeta(x)^2\ dx - 2t^2\int_\Omega \nabla u^t(x)u^t(x)\cdot\zeta(x)\nabla\zeta(x)\ dx\\ &\leq \varepsilon t^2\int_\Omega |u^t(x)|^2|\nabla\zeta(x)|^2\ dx + \frac{t^2}{\varepsilon}\int_\Omega |\nabla u^t(x)|^2\ dx.\end{aligned}\tag{3.4}$$

Combining (3.4) and

$$|\nabla\zeta(x)| = \alpha|\nabla\eta(x)|e^{\alpha\eta(x)} \leq \frac{1}{t}e^{\alpha\eta(x)},$$

we get

$$\int_\Omega |u^t(x)|^2\zeta(x)^2\ dx \leq \varepsilon\int_\Omega |u^t(x)|^2e^{2\alpha\eta(x)}\ dx + \frac{t^2}{\varepsilon}\int_\Omega |\nabla u^t(x)|^2\ dx. \tag{3.5}$$

For the first term in (3.5), we have

$$\begin{aligned}\int_\Omega |u^t(x)|^2e^{2\alpha\eta(x)}\ dx &\leq \int_\Omega |u^t(x)|^2\zeta(x)^2\ dx + 2\int_\Omega |u^t(x)|^2e^{\alpha\eta(x)}\ dx\\ &\leq \int_\Omega |u^t(x)|^2\zeta(x)^2\ dx + 2e^\alpha\int_\Omega |f(x)|^2\ dx\end{aligned}\tag{3.6}$$

Meanwhile, by virtue of Lemma 2.14, we have

$$\int_\Omega |\nabla u^t(x)|^2\, dx \le \frac{1}{t^2}\|f\|_{L^2(E)}^2. \tag{3.7}$$

Combining (3.5)-(3.7), we get

$$\int_\Omega |u^t(x)|^2\zeta(x)^2\, dx \le \varepsilon \int_\Omega |u^t(x)|^2\zeta(x)^2\, dx + \left(2\varepsilon + \frac{1}{e^2\varepsilon}\right) e^\alpha \|f\|_{L^2(E)}^2. \tag{3.8}$$

Taking $\varepsilon = \frac{1}{2}$, we get

$$\int_\Omega |u^t(x)|^2\zeta(x)^2\, dx \lesssim e^\alpha \|f\|_{L^2(E)}^2. \tag{3.9}$$

Finally, we combine (3.2) and (3.9) to obtain

$$\|u^t\|_{L^2(F)} \lesssim e^{-\frac{\alpha}{2}}\|f\|_{L^2(E)} \le e^{-\frac{ad}{t}}\|f\|_{L^2(E)},$$

where $a = \frac{1}{2C}$. This completes the proof of (3.1). □

By using Proposition 3.1 and the Riesz-Thorin interpolation theorem, we obtain a Gaffney type estimate for $(1-\nu\Delta)^{-1}$ for every $p \in (1,\infty)$.

Theorem 3.2. *Let $\nu > 0$ be a constant, Ω be a bounded smooth domain in $\mathbb{R}^n$, and $1 < p < \infty$. Let E and F be disjoint closed subsets of Ω with $d := \mathrm{dist}(E, F) > 0$. Then there exist some constants $C > 0$ and $a = a_p > 0$ such that for all $f \in L^p(\Omega)$ satisfying* $\mathrm{supp}(f) \subset E$, *we have*

$$\|(1-\nu\Delta)^{-1}f\|_{L^p(F)} \le C \exp\left(-a\frac{d}{\sqrt{\nu}}\right)\|f\|_{L^p(E)}. \tag{3.10}$$

Proof. Let $1 < p < 2$. We know that $(1-\nu\Delta)^{-1}$ is bounded on $L^1(\Omega)$, while $(1-\nu\Delta)^{-1}$ is subject to a strong decay estimate (3.1) in $L^2(\Omega)$, so we can interpolate these estimates to obtain

$$\|(1-\nu\Delta)^{-1}f\|_{L^p(F)} \lesssim e^{-\frac{2a(p-1)d}{p\sqrt{\nu}}}\|f\|_{L^p(E)}.$$

For the case $p > 2$, we can interpolate the boundedness of $(1-\nu\Delta)^{-1}$ on $L^\infty(\Omega)$ and the inequality (3.1) to obtain (3.10). □

§ 4. The proof of Theorem 1.2

First we prove the following pointwise estimates by utilizing Theorem 3.2.

Lemma 4.1. *Let $n > 2$ and $1 < p < \infty$. Then for every $f \in L^p(\Omega)$ and $1 < q < p$, we have*

$$M((1-\nu\Delta)^{-1}f)(x) \lesssim M(|f|^q)(x)^{1/q}, \ (x \in \Omega).$$

Proof. Extend f to be zero outside Ω. Let $x \in \Omega$ and $r > 0$. We decompose

$$f = \sum_{k=0}^{\infty} f_k,$$

where

$$f_0 := f\chi_{B(x,2r)} \text{ and } f_k := f\chi_{B(x,2^{k+1}r)\setminus B(x,2^k r)}.$$

Let $u := (1-\nu\Delta)^{-1}f$ and $u_k := (1-\nu\Delta)^{-1}f_k$ for $k \in \mathbb{N} \cup \{0\}$. Then

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |u(y)|\, dy \leq \sum_{k=0}^{\infty} \frac{1}{|B(x,r)|}\int_{B(x,r)} |u_k(y)|\, dy. \tag{4.1}$$

As a consequence of (1.3) and Hölder's inequality, we have

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |u_0(y)|\, dy \lesssim \frac{1}{|B(x,r)|}\int_{B(x,2r)} |f(y)|\, dy \lesssim M(|f|^q)(x)^{\frac{1}{q}}. \tag{4.2}$$

In view of (4.1) and (4.2), it remains to show that

$$\sum_{k=1}^{\infty} \frac{1}{|B(x,r)|}\int_{B(x,r)} |u_k(y)|\, dy \lesssim M(|f|^q)(x)^{\frac{1}{q}}. \tag{4.3}$$

By virtue of Fubini's theorem and Lemma 2.13, we have

$$\int_{B(x,r)} |u_k(y)|\, dy \lesssim e^{-C2^k r}\int_{B(x,2^{k+1}r)\setminus B(x,2^k r)} |f(z)| I_2\chi_{B(x,r)}(z)\, dz. \tag{4.4}$$

Hence, by combining (4.4) and Lemma 2.7, we get

$$\int_{B(x,r)} |u_k(y)|\, dy \lesssim r^2 e^{-C2^k r}\int_{B(x,2^{k+1}r)\setminus B(x,2^k r)} |f(z)| M\chi_{B(x,r)}(z)^{\frac{n-2}{n}}\, dz. \tag{4.5}$$

Observe that

$$M\chi_{B(x,r)}(y) \sim (1 + r^{-1}|x-y|)^{-n}$$

for $y \in \mathbb{R}^n$. Therefore, by applying this observation to (4.5), we get

$$\begin{aligned}\int_{B(x,r)} |u_k(y)|\, dy &\lesssim r^2 e^{-C2^k r}\int_{B(x,2^{k+1}r)\setminus B(x,2^k r)} \frac{|f(z)|}{(1+r^{-1}|x-z|)^{n-2}}\, dz \\ &\leq \frac{r^2 e^{-C2^k r}}{2^{k(n-2)}}\int_{B(x,2^{k+1}r)} |f(z)|\, dz. \end{aligned} \tag{4.6}$$

By Hölder's inequality, we have

$$\int_{B(x,2^{k+1}r)} |f(z)|\, dz \le |B(x,2^{k+1}r)| M(|f|^q)(x)^{\frac{1}{q}}. \tag{4.7}$$

Combining (4.6) and (4.7), we get

$$\sum_{k=1}^{\infty} \frac{1}{|B(x,r)|} \int_{B(x,r)} |u_k(y)|\, dy \lesssim M(|f|^q)(x)^{\frac{1}{q}} \sum_{k=1}^{\infty} (2^k r)^2 e^{-C2^k r}.$$

Since

$$\begin{aligned}\sum_{k=1}^{\infty} (2^k r)^2 e^{-C2^k r} &= \frac{2}{3} \sum_{k=1}^{\infty} e^{-C2^k r} \int_{2^k r}^{2^{k+1} r} t\, dt \\ &\lesssim \sum_{k=1}^{\infty} \int_{2^k r}^{2^{k+1} r} t e^{-\frac{Ct}{2}}\, dt \\ &= \int_{2r}^{\infty} t e^{-\frac{Ct}{2}}\, dt \\ &\le \int_{0}^{\infty} t e^{-\frac{Ct}{2}}\, dt < \infty,\end{aligned}$$

independent of $r > 0$, we see that

$$\sum_{k=1}^{\infty} \frac{1}{|B(x,r)|} \int_{B(x,r)} |u_k(y)|\, dy \lesssim M(|f|^q)(x)^{\frac{1}{q}},$$

as desired. □

Finally, we prove Theorem 1.2 by combining Lemma 4.1 and Theorem 2.8.

Proof of Theorem 1.2. Let $u := (1 - \nu\Delta)^{-1} f$ and let $x \in \Omega$. Then, by virtue of Lemma 4.1, we have

$$|u(x)| \le Mu(x) \lesssim M(|f|^q)(x)^{\frac{1}{q}},$$

for every $1 < q < p$. Since $\frac{p}{q} > 1$ and $\varphi^q \in \mathcal{G}_{p/q}$, we may use Theorem 2.8 to obtain

$$\begin{aligned}\|u\|_{\mathcal{M}_p^{\varphi}(\Omega)} &\lesssim \|M(|f|^q)^{\frac{1}{q}}\|_{\mathcal{M}_p^{\varphi}(\Omega)} \\ &= \|M(|f|^q)\|_{\mathcal{M}_{p/q}^{\varphi^q}(\Omega)}^{1/q} \\ &\lesssim \||f|^q\|_{\mathcal{M}_{p/q}^{\varphi^q}(\Omega)}^{1/q} = \|f\|_{\mathcal{M}_p^{\varphi}(\Omega)},\end{aligned}$$

as required. □

§ 5. The proof of Theorems 1.3 and 1.4

§ 5.1. A Gaffney type estimate for $(1+\nu L)^{-1}$ in Lebesgue spaces

First let us recall the following Gaffney type estimates in $L^2(\Omega)$.

Lemma 5.1. [1, Lemma 2.1] *Let Ω be a bounded smooth domain in $\mathbb{R}^n$. Let E and F be two closed subsets of Ω with $d := \operatorname{dist}(E,F) > 0$. Then there exists a positive constant C such that*

$$\int_F |D((1+\nu L)^{-1} f)(x)|^2 \, dx \lesssim e^{-\frac{d}{C\sqrt{\nu}}} \int_E |f(x)|^2 \, dx, \tag{5.1}$$

for every $f \in L^2(\Omega)$ with $\operatorname{supp}(f) \subseteq E$.

For the case $p > 2$, we interpolate Lemma 5.1 with the following result.

Proposition 5.2. [5, Theorem 4.7] *Let $n \geq 3$ and Ω be a bounded smooth domain in $\mathbb{R}^n$. Assume that $a_{ij} \in \mathrm{VMO}(\Omega)$. Let $2 < p < \infty$, $\frac{1}{p_*} := \frac{1}{p} + \frac{1}{n}$, and $f \in L^{p_*}(\Omega)$. Then the following inequality*

$$\|D((1+\nu L)^{-1} f)\|_{L^p(\Omega)} \lesssim \|f\|_{L^{p_*}(\Omega)} \tag{5.2}$$

holds.

Now we prove an extension of Lemma 5.1 for $p > 2$.

Lemma 5.3. *Let $n \geq 2$, Ω be a bounded smooth domain in $\mathbb{R}^n$, $\nu > 0$, and $2 < p < \infty$. Assume that $a_{ij} \in \mathrm{VMO}(\Omega)$. Suppose that E and F are two closed subsets of Ω with $d := \operatorname{dist}(E,F) > 0$. Then there exists a constant $C > 0$ such that*

$$\|D((1+\nu L)^{-1} f)\|_{L^p(F)} \lesssim e^{-\frac{d}{C\sqrt{\nu}}} \|f\|_{L^p(E)}, \tag{5.3}$$

for every $f \in L^p(\Omega)$ with $\operatorname{supp}(f) \subseteq E$.

Proof. Let $u := (1+\nu L)^{-1} f$ and define p^* by $\frac{1}{p^*} := \frac{1}{p} - \frac{1}{n}$. Observe that $\theta := \frac{\frac{1}{2}-\frac{1}{p}}{\frac{1}{2}-\frac{1}{p^*}}$ satisfies $0 < \theta < 1$ and

$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{p^*}.$$

Therefore, by Hölder's inequality, we get

$$\|Du\|_{L^p(F)} \leq \|Du\|_{L^2(F)}^{1-\theta} \|Du\|_{L^{p^*}(F)}^{\theta}.$$

Combining this inequality with (5.1), (5.2), and the embedding $L^p(E) \subseteq L^2(E)$, we get (5.3), as desired. □

§ 5.2. The proof of Theorem 1.3

Set f to be zero outside Ω. Let $x \in \Omega$ and $0 < r < \mathrm{diam}(\Omega)$. We shall show that

$$\frac{\varphi(r)}{|B(x,r)|^{1/2}} \left(\int_{B(x,r)\cap\Omega} |D((1+\nu L)^{-1} f)(y)|^2 \, dy \right)^{\frac{1}{2}} \lesssim \|f\|_{\mathcal{M}_2^\varphi(\Omega)}. \tag{5.4}$$

We define

$$f_0 := f\chi_{B(x,2r)} \quad \text{and} \quad f_k := f\chi_{B(x,2^{k+1}r)\setminus B(x,2^k r)}, \ k \in \mathbb{N}.$$

For $k \in \mathbb{N} \cup \{0\}$, we define $u_k := (1+\nu L)^{-1} f_k$. Then

$$\left(\int_{B(x,r)\cap\Omega} |D((1+\nu L)^{-1} f)(y)|^2 \, dy \right)^{\frac{1}{2}} \le \sum_{k=0}^{\infty} \left(\int_{B(x,r)\cap\Omega} |Du_k(y)|^2 \, dy \right)^{1/2}. \tag{5.5}$$

It follows from (1.8) that

$$\begin{aligned} \left(\int_{B(x,r)\cap\Omega} |Du_0(y)|^2 \, dy \right)^{1/2} &\le \left(\int_{B(x,r)\cap\Omega} |Du_0(y)|^2 \, dy \right)^{1/2} \\ &\lesssim \left(\int_{B(x,2r)\cap\Omega} |f(y)|^2 \, dy \right)^{1/2} \\ &\lesssim \frac{|B(x,r)|^{1/2}}{\varphi(r)} \|f\|_{\mathcal{M}_2^\varphi(\Omega)}. \end{aligned} \tag{5.6}$$

By virtue of Lemma 5.1, for $k \in \mathbb{N}$, we have

$$\begin{aligned} \left(\int_{B(x,r)\cap\Omega} |Du_k(y)|^2 \, dy \right)^{\frac{1}{2}} &\lesssim e^{-\frac{C(2^k-1)r}{\sqrt{nu}}} \left(\int_{\Omega} |f_k(y)|^2 \, dy \right)^{\frac{1}{2}} \\ &\lesssim \|f\|_{\mathcal{M}_2^\varphi(\Omega)} e^{-C2^k r} \frac{|B(x,2^{k+1}r|^{\frac{1}{2}}}{\varphi(2^{k+1}r)}. \end{aligned} \tag{5.7}$$

Since

$$\begin{aligned} \sum_{k=1}^{\infty} e^{-C2^k r} \frac{|B(x,2^{k+1}r|^{\frac{1}{2}}}{\varphi(2^{k+1}r)} &\lesssim \sum_{k=1}^{\infty} \int_{C2^k r}^{C2^{k+1}r} e^{-t} \frac{t^{n/2-1}}{\varphi(t)} \, dt \\ &= \int_{2Cr}^{\infty} e^{-t} \frac{t^{n/2-1}}{\varphi(t)} \, dt \\ &\lesssim \frac{r^{n/2}}{\varphi(r)}, \end{aligned} \tag{5.8}$$

we see that

$$\sum_{k=1}^{\infty} \left(\int_{B(x,r)\cap\Omega} |Du_k(y)|^2 \, dy \right)^{\frac{1}{2}} \lesssim \frac{r^{n/2}}{\varphi(r)} \|f\|_{\mathcal{M}_2^\varphi(\Omega)}. \tag{5.9}$$

Combining (5.5), (5.6), and (5.9), we get (5.4), as desired.

§ 5.3. The proof of Theorem 1.4

The proof of Theorem 1.4 is similar to that of Theorem 1.3. The difference is that we now apply Proposition 5.2 and Lemma 5.3. Let $x \in \Omega$ and $0 < r < \operatorname{diam}(\Omega)$. Let f_k and u_k be defined as in the proof of Theorem 1.3. By virtue of Proposition 5.2 and the Hölder inequality, we have

$$\begin{aligned}\left(\int_{B(x,r)\cap\Omega} |Du_0(y)|^p \, dy\right)^{1/p} &\lesssim |\Omega|^{\frac{1}{n}} \left(\int_{B(x,2r)\cap\Omega} |f(y)|^p \, dy\right)^{1/p} \\ &\lesssim \frac{|B(x,r)|^{1/p}}{\varphi(r)} \|f\|_{\mathcal{M}_p^\varphi(\Omega)}\end{aligned} \tag{5.10}$$

Now for $k \in \mathbb{N}$, we apply Lemma 5.3 to obtain

$$\left(\int_{B(x,r)\cap\Omega} |Du_k(y)|^p \, dy\right)^{\frac{1}{p}} \lesssim \|f\|_{\mathcal{M}_p^\varphi(\Omega)} e^{-C2^k r} \frac{|B(x,2^{k+1}r|^{\frac{1}{p}}}{\varphi(2^{k+1}r)}. \tag{5.11}$$

As a consequence of (1.11), we get

$$\sum_{k=1}^{\infty} e^{-C2^k r} \frac{|B(x,2^{k+1}r|^{\frac{1}{p}}}{\varphi(2^{k+1}r)} \lesssim \int_{2Cr}^{\infty} e^{-t} \frac{t^{n/p-1}}{\varphi(t)} \, dt \lesssim \frac{r^{n/p}}{\varphi(r)}. \tag{5.12}$$

Therefore,

$$\sum_{k=1}^{\infty} \left(\int_{B(x,r)\cap\Omega} |Du_k(y)|^p \, dy\right)^{\frac{1}{p}} \lesssim \frac{r^{n/p}}{\varphi(r)} \|f\|_{\mathcal{M}_p^\varphi(\Omega)}. \tag{5.13}$$

Combining (5.10) and (5.13), we get

$$\begin{aligned}&\frac{\varphi(r)}{|B(x,r)|^{\frac{1}{p}}} \left(\int_{B(x,r)\cap\Omega} |D((1+\nu L)^{-1} f)(y)|^p \, dy\right)^{\frac{1}{p}} \\ &\leq \frac{\varphi(r)}{|B(x,r)|^{\frac{1}{p}}} \sum_{k=0}^{\infty} \left(\int_{B(x,r)\cap\Omega} |Du_k(y)|^p \, dy\right)^{1/p} \lesssim \|f\|_{\mathcal{M}_p^\varphi(\Omega)},\end{aligned} \tag{5.14}$$

as desired.

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